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# Loewner matrices of matrix convex and monotone functions : joint work with F.Hiai (Noncommutative Structure in Operator Theory and its Application)

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Loewner matrices of matrix convex and monotone functions  
(joint work with F. Hiai)

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Some results in [3, 6] were reported. Here we collect results from them. For the detail, please see the papers.

## 1 Characterisations by Bhatia-Sano

In this section, we consider a  $C^1$  function  $f$  from the interval  $(0, \infty)$  into itself, with  $f(0) = \lim_{t \rightarrow 0^+} f(t) = 0$ . Given any  $n$  distinct points  $p_1, \dots, p_n$  in  $(0, \infty)$ , let  $L_f(p_1, \dots, p_n)$  be the  $n \times n$  matrix defined as

$$L_f(p_1, \dots, p_n) = \left[ \frac{f(p_i) - f(p_j)}{p_i - p_j} \right]. \quad (1.1)$$

When  $i = j$  the quotient in (1.1) is interpreted as  $f'(p_i)$ . Such a matrix is called a *Loewner matrix* associated with  $f$ .

For the function  $f(t) = t^r$  where  $r > 0$ , we use the symbol  $L_r$  for a Loewner matrix associated with this function. Thus

$$L_r = \left[ \frac{p_i^r - p_j^r}{p_i - p_j} \right]. \quad (1.2)$$

The function  $f$  is said to be *operator monotone* on  $[0, \infty)$  if for two positive semidefinite matrices  $A$  and  $B$  (of any size  $n$ ) the inequality  $A \geq B$  implies  $f(A) \geq f(B)$ . Here, as usual,  $A \geq B$  means that  $A - B$  is positive semidefinite (p.s.d. for short).

Karl Löwner (later Charles Loewner) in [9] showed that  $f$  is operator monotone if and only if for all  $n$ , and all  $p_1, \dots, p_n$ , the Loewner matrices  $L_f(p_1, \dots, p_n)$  are p.s.d. and that the function  $f(t) = t^r$  is operator monotone if and only if  $0 < r \leq 1$ . Consequently, if  $0 < r \leq 1$ , then the matrix (1.2) is p.s.d., and therefore all its eigenvalues are non-negative.

Recall the notion of operator convexity: Assume that  $f$  is a  $C^2$  function from  $(0, \infty)$  into itself,  $f(0) = 0$  and  $f'(0) = 0$ . We say that  $f$  is *operator convex* if

$$f((1-t)A + tB) \leq (1-t)f(A) + tf(B), \quad 0 \leq t \leq 1,$$

for all p.s.d. matrices  $A$  and  $B$  (of any size  $n$ ).

Let  $H^n$  be the subspace of  $\mathbb{C}^n$  consisting of all  $x = (x_1, \dots, x_n)$  for which  $\sum_{i=1}^n x_i = 0$ . An  $n \times n$  Hermitian matrix  $A$  is said to be *conditionally positive definite* (c.p.d. for short) or *almost positive* if

$$\langle x, Ax \rangle \geq 0 \quad \text{for all } x \in H^n,$$

and *conditionally negative definite* (c.n.d. for short) if  $-A$  is c.p.d. We refer the reader to [1, 4, 8] for properties of these matrices.

We proved:

**Theorem 1.1.** Let  $f$  be an operator convex function. Then all Loewner matrices associated with  $f$  are conditionally negative definite.

**Theorem 1.2.** Let  $f(t) = tg(t)$  where  $g$  is an operator convex function. Then all Loewner matrices associated with  $f$  are conditionally positive definite.

**Theorem 1.3.** Let  $L_r$  be the  $n \times n$  Loewner matrix (1.2) associated with distinct points  $p_1, \dots, p_n$ . Then

- (i)  $L_r$  is conditionally negative definite for  $1 \leq r \leq 2$ , and conditionally positive definite for  $2 \leq r \leq 3$ .
- (ii)  $L_r$  is nonsingular for  $1 < r < 2$  and for  $2 < r < 3$ .
- (iii) As a consequence, for  $1 < r < 2$  the matrix  $L_r$  has one positive and  $n - 1$  negative eigenvalues, and for  $2 < r < 3$  it has one negative and  $n - 1$  positive eigenvalues.

Here is the converse of Theorems 1.1 and 1.2:

**Theorem 1.4.** Let  $f$  be a  $C^2$  function from  $(0, \infty)$  into itself with  $f(0) = f'(0) = 0$ . Suppose all Loewner matrices  $L_f$  are conditionally negative definite. Then  $f$  is operator convex.

**Theorem 1.5.** Let  $f$  be a  $C^3$  function from  $(0, \infty)$  into itself with  $f(0) = f'(0) = f''(0) = 0$ . Suppose all Loewner matrices  $L_f$  are conditionally positive definite. Then there exists an operator convex function  $g$  such that  $f(t) = tg(t)$ .

**Remark.** Theorems 1.1, 1.2, 1.4 and 1.5 together say the following. Let  $f$  be a  $C^3$  function from  $(0, \infty)$  into itself with  $f(0) = 0$ . Let  $g(t) = tf(t)$ ,  $h(t) = t^2f(t)$ . Then the following three conditions are equivalent.

- (i) All Loewner matrices  $L_f$  are p.s.d.
- (ii) All Loewner matrices  $L_g$  are c.n.d.
- (iii) All Loewner matrices  $L_h$  are c.p.d.

## 2 Generalisations by Hiai-Sano

We already review characterizations in [3] for operator convexity of nonnegative functions on  $[0, \infty)$  in terms of the conditional negative or positive definiteness of the Loewner matrices. Uchiyama [10] extended, by a rather different method, results in such a way that the assumption  $f \geq 0$  is removed and the boundary condition  $f(0) = f'(0) = 0$  is relaxed. Note that the conditional positive definiteness of the Loewner matrices and the matrix/operator monotony were related in [7] and [4, Chapter XV] for a real function on a general open interval.

We proved:

**Theorem 2.1.** *Let  $f$  be a real  $C^1$  function on  $(0, \infty)$ . For each  $n \in \mathbb{N}$  consider the following conditions:*

- (a)<sub>n</sub>  $f$  is  $n$ -convex on  $(0, \infty)$ ;
- (b)<sub>n</sub>  $\liminf_{t \rightarrow \infty} f(t)/t > -\infty$  and  $L_f(t_1, \dots, t_n)$  is c.n.d. for all  $t_1, \dots, t_n \in (0, \infty)$ ;
- (c)<sub>n</sub>  $\limsup_{t \searrow 0} tf(t) \geq 0$  and  $L_{tf(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $t_1, \dots, t_n \in (0, \infty)$ .

Then for every  $n \in \mathbb{N}$  the following implications hold:

$$(a)_{2n+1} \implies (b)_n, \quad (b)_{4n+1} \implies (a)_n, \quad (a)_{n+1} \implies (c)_n, \quad (c)_{2n+1} \implies (a)_n.$$

**Corollary 2.2.** *Let  $f$  be a real  $C^1$  function on  $(0, \infty)$ . Then the following conditions are equivalent:*

- (a)  $f$  is operator convex on  $(0, \infty)$ ;
- (b)  $\liminf_{t \rightarrow \infty} f(t)/t > -\infty$  and  $L_f(t_1, \dots, t_n)$  is c.n.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (0, \infty)$ ;
- (c)  $\limsup_{t \searrow 0} tf(t) \geq 0$  and  $L_{tf(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (0, \infty)$ .

Moreover, if the above conditions are satisfied, then  $\lim_{t \rightarrow \infty} f(t)/t$  and  $\lim_{t \searrow 0} tf(t)$  exist in  $(-\infty, \infty]$  and  $[0, \infty)$ , respectively.

**Theorem 2.3.** *Let  $f$  be a real  $C^1$  function on  $(0, \infty)$ . For each  $n \in \mathbb{N}$  consider the following conditions:*

- (a)'<sub>n</sub>  $f$  is  $n$ -monotone on  $(0, \infty)$ ;
- (b)'<sub>n</sub>  $\limsup_{t \rightarrow \infty} f(t)/t < +\infty$ ,  $\limsup_{t \rightarrow \infty} f(t) > -\infty$ , and  $L_f(t_1, \dots, t_n)$  is c.p.d. for all  $t_1, \dots, t_n \in (0, \infty)$ ;
- (c)'<sub>n</sub>  $\liminf_{t \searrow 0} tf(t) \leq 0$ ,  $\limsup_{t \rightarrow \infty} f(t) > -\infty$ , and  $L_{tf(t)}(t_1, \dots, t_n)$  is c.n.d. for all  $t_1, \dots, t_n \in (0, \infty)$ ;
- (d)'<sub>n</sub>  $\liminf_{t \searrow 0} tf(t) \leq 0$ ,  $\limsup_{t \searrow 0} t^2 f(t) \geq 0$ , and  $L_{t^2 f(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $t_1, \dots, t_n \in (0, \infty)$ .

Then for every  $n \in \mathbb{N}$  the following implications hold:

$$\begin{aligned} (a)'_n &\implies (b)'_n \text{ if } n \geq 2, & (b)'_{4n+1} &\implies (a)'_n, & (a)'_{2n+2} &\implies (c)'_n, & (c)'_{2n+1} &\implies (a)'_n, \\ (a)'_n &\implies (d)'_n \text{ if } n \geq 2, & (c)'_{2n+1} &\implies (d)'_n, & (d)'_{2n+1} &\implies (c)'_n. \end{aligned}$$

**Corollary 2.4.** Let  $f$  be a real  $C^1$  function on  $(0, \infty)$ . Then the following conditions are equivalent:

- (a)'  $f$  is operator monotone on  $(0, \infty)$ ;
- (b)'  $\limsup_{t \rightarrow \infty} f(t)/t < +\infty$ ,  $\limsup_{t \rightarrow \infty} f(t) > -\infty$ , and  $L_f(t_1, \dots, t_n)$  is c.p.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (0, \infty)$ ;
- (c)'  $\liminf_{t \searrow 0} tf(t) \leq 0$ ,  $\limsup_{t \rightarrow \infty} f(t) > -\infty$ , and  $L_{tf(t)}(t_1, \dots, t_n)$  is c.n.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (0, \infty)$ ;
- (d)'  $\liminf_{t \searrow 0} tf(t) \leq 0$ ,  $\limsup_{t \searrow 0} t^2 f(t) \geq 0$ , and  $L_{t^2 f(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (0, \infty)$ .

Moreover, if the above conditions are satisfied, then  $\lim_{t \rightarrow \infty} f(t)/t$ ,  $\lim_{t \rightarrow \infty} f(t)$ , and  $\lim_{t \searrow 0} tf(t)$  exist in  $[0, \infty)$ ,  $(-\infty, \infty]$ , and  $(-\infty, 0]$ , respectively, and  $\lim_{t \searrow 0} t^\alpha f(t) = 0$  for any  $\alpha > 1$ .

**Proposition 2.5.** Consider the power functions  $t^\alpha$  on  $(0, \infty)$ , where  $\alpha \in \mathbb{R}$ . Then:

- (1)  $t^\alpha$  is 2-monotone if and only if  $0 \leq \alpha \leq 1$ , or equivalently,  $t^\alpha$  is operator monotone. Moreover,  $-t^\alpha$  is 2-monotone if and only if  $-1 \leq \alpha \leq 0$ .
- (2)  $t^\alpha$  is 2-convex if and only if either  $-1 \leq \alpha \leq 0$  or  $1 \leq \alpha \leq 2$ , or equivalently,  $t^\alpha$  is operator convex.
- (3)  $L_{t^\alpha}(t_1, t_2)$  is c.p.d. for all  $t_1, t_2 \in (0, \infty)$  if and only if either  $0 \leq \alpha \leq 1$  or  $\alpha \geq 2$ .
- (4)  $L_{t^\alpha}(t_1, t_2)$  is c.n.d. for all  $t_1, t_2 \in (0, \infty)$  if and only if either  $\alpha \leq 0$  or  $1 \leq \alpha \leq 2$ .

(5)  $L_{t^\alpha}(t_1, t_2, t_3)$  is c.p.d. for all  $t_1, t_2, t_3 \in (0, \infty)$  if and only if either  $0 \leq \alpha \leq 1$  or  $2 \leq \alpha \leq 3$ .

(6)  $L_{t^\alpha}(t_1, t_2, t_3)$  is c.n.d. for all  $t_1, t_2, t_3 \in (0, \infty)$  if and only if either  $-1 \leq \alpha \leq 0$  or  $1 \leq \alpha \leq 2$ .

**Theorem 2.6.** Let  $f$  be a real  $C^1$  function on  $(a, b)$  where  $-\infty < a < b < \infty$ . For each  $n \in \mathbb{N}$  consider the following conditions:

$(\alpha)_n$   $f$  is  $n$ -monotone on  $(a, b)$ ;

$(\beta)_n$   $\limsup_{t \nearrow b} (b-t)f(t) < +\infty$ ,  $\limsup_{t \nearrow b} f(t) > -\infty$ , and  $L_{(b-t)^2 f(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $t_1, \dots, t_n \in (a, b)$ ;

$(\gamma)_n$   $\liminf_{t \searrow a} (t-a)f(t) \leq 0$ ,  $\limsup_{t \nearrow b} f(t) > -\infty$ , and  $L_{(t-a)(b-t)f(t)}(t_1, \dots, t_n)$  is c.n.d. for all  $t_1, \dots, t_n \in (a, b)$ ;

$(\delta)_n$   $\liminf_{t \searrow a} (t-a)f(t) \leq 0$ ,  $\limsup_{t \searrow a} (t-a)^2 f(t) \geq 0$ , and  $L_{(t-a)^2 f(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $t_1, \dots, t_n \in (a, b)$ .

Then for every  $n \in \mathbb{N}$  the following implications hold:

$$(\alpha)_n \implies (\beta)_n \text{ if } n \geq 2, \quad (\beta)_{4n+1} \implies (\alpha)_n, \quad (\alpha)_{2n+2} \implies (\gamma)_n, \quad (\gamma)_{2n+1} \implies (\alpha)_n,$$

$$(\alpha)_n \implies (\delta)_n \text{ if } n \geq 2, \quad (\gamma)_{2n+1} \implies (\delta)_n, \quad (\delta)_{2n+1} \implies (\gamma)_n.$$

**Corollary 2.7.** Let  $f$  be a real  $C^1$  function on  $(a, b)$  where  $-\infty < a < b < \infty$ . Then the following conditions are equivalent:

$(\alpha)$   $f$  is operator monotone on  $(a, b)$ ;

$(\beta)$   $\limsup_{t \nearrow b} (b-t)f(t) < +\infty$ ,  $\limsup_{t \nearrow b} f(t) > -\infty$ , and  $L_{(b-t)^2 f(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (a, b)$ ;

$(\gamma)$   $\liminf_{t \searrow a} (t-a)f(t) \leq 0$ ,  $\limsup_{t \nearrow b} f(t) > -\infty$ , and  $L_{(t-a)(b-t)f(t)}(t_1, \dots, t_n)$  is c.n.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (a, b)$ ;

$(\delta)$   $\liminf_{t \searrow a} (t-a)f(t) \leq 0$ ,  $\limsup_{t \searrow a} (t-a)^2 f(t) \geq 0$ , and  $L_{(t-a)^2 f(t)}(t_1, \dots, t_n)$  is c.p.d. for all  $n \in \mathbb{N}$  and all  $t_1, \dots, t_n \in (a, b)$ .

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